THE INTERIOR FIELD OF A MAGNETIZED EINSTEIN-MAXWELL OBJECT

Tonatiuh Matos*,
Departamento de Física,
Centro de Investigación y de Estudios Avanzados del IPN,
P.O. Box 14-740, México 07000, D.F., México.
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Using the Harmonic map ansatz, we reduce the axisymmetric, static Einstein-Maxwell equations coupled with a magnetized perfect fluid to a set of Poisson-like equations. We were able to integrate the Poisson equations in terms of an arbitrary function $M = M(\rho, \zeta)$ and some integration constants. The thermodynamic equation restricts the solutions to only some state equations, but in some cases when the solution exists, the interior solution can be matched with the corresponding exterior one.

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Exact solutions of the Einstein's field equations have been one of the most interesting challenge for mathematicians and physicist [1]. A great effort has been done for finding exact solutions with a physical interpretation. In the seventies mathematicians and relativists had a great success using very elaborate mathematical technics, a great amount of exact solutions of the Einstein equations have been therefore found, analyzed and studied in the seventies and eighties. The discovery of the binary pulsars gave rise to non-perturbative effects of general relativity and the exact solutions of the Einstein's field equations become a necessarily subject for astrophysicists and relativists. Furthermore, for compact objects like white dwarfs, pulsars and black holes, non-pertubative effects are the most interesting one, effects which can only be understood better with exact solutions of the field equations.

In this letter I am interested in the interior field of an object, taking matter into account. I will suppose axial symmetry only, and allow metric functions with as most freedom as we can. As a first approximation I will suppose that the metric is static. Now we must determine the right hand side of the Einstein equations. To describe this object one can start modeling its interior matter as a perfect fluid. It is difficult to know the state equation of a compact object, we do not use to work with matter in so extreme conditions of density and pressure. We even do not know if the energy conditions for the energy momentum tensor are valid at this extreme. Something we can do, is to make a general ansatz about the state equation, for example $p = \omega\mu$, where ω is a constant, p is the pressure of matter and μ is its energy density, and let that the theory says us something about this ansatz. As a first approximation this ansatz seems to be reasonable, but to have a more complete study of this class of matter, we must investigate other possibilities.

Let me start with this ansatz and investigate what the Einstein-Maxwell theory can say about it. The action we deal with is then

$$\mathcal{S} = \int d^4x\sqrt{-g}[-R + F^{\mu\nu}F_{\mu\nu} + \mathcal{L}_{Matter}] \tag{1}$$

where $g = \det(g_{\mu\nu})$, $\mu, \nu = 0, 1, 2, 3$, R is the scalar curvature and $F_{\mu\nu}$ is the Maxwell tensor.

After variation with respect to the electromagnetic field and the metric, we respectively obtain the following field equations

$$F^{\mu\nu}{}_{;\nu} = 0,$$
$$G_{\mu\nu} = 2(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) + 2T_{pf\,\mu\nu}\ ,$$
$$T_{pf}{}^{\mu\nu}{}_{;\nu} = 0 \tag{2}$$

where the last field equation in (2) is the thermodynamic one. In this work we will only consider the axisymmetric static case, that means, space-times containing one space-like and one time-like commuting, hypersurface forming Killing vector, and the case when matter corresponds to a perfect fluid, *i.e.* $T_{pf}{}^{\mu}{}_{\nu}$ =diag$(p, p, p, -\mu)$. The axisymmetric static metric convenient for the energy conditions of this case is the static Papapetrou metric [1]

$$ds^2 = \frac{1}{f}\{e^{2k}\ (d\rho^2 + d\zeta^2) + W^2 d\varphi^2\} - f\ dt^2 \tag{3}$$

where f, W, and k are functions of ρ and ζ only. Now we use the harmonic map ansatz. Suppose that λ and τ are coordinates of a two dimensional flat space $ds^2_{v_p} = d\lambda\, d\tau$, with $\lambda = \lambda(\rho, \zeta)$ and $\tau = \tau(\rho, \zeta)$ and suppose that $f = f(\lambda,\ \tau)$. Using the procedure of reference [2,3] we find

that $f = e^{\lambda-\gamma\tau}$ is a good parametrization for a space-time curved by a gravitational and an electromagnetic field. In terms of the functions λ and τ the axisymmetric static Einstein field equations for the action (1) read

$$2\Delta\lambda = (3p+\mu)\sqrt{-g}$$

$$\Delta \ln W = 2p\sqrt{-g}$$

$$\Delta\tau + 2\ W((\tau_{,\rho})^2 + (\tau_{,\zeta})^2)e^{\lambda-(\gamma-2)\tau} = 0 \qquad (4)$$

where $\Delta f = \frac{1}{2}[(Wf_{,\rho})_{,\rho} + (Wf_{,\zeta})_{,\zeta}]$ is the generalized Laplace operator. The electromagnetic four potential is given as $(A_\rho, A_\zeta, A_\varphi, A_t) = (0, 0, A_\varphi, 0)$. In terms of the function τ, the Maxwell equations read

$$A_{\varphi,\rho} = Q\ We^\tau\tau_{,\zeta}, \quad A_{\varphi,\zeta} = -Q\ We^\tau\tau_{,\rho}, \qquad (5)$$

where $2Q^2 = \gamma$. Observe that the integrability condition for the electromagnetic function A_φ given in (5) implies that $\Delta e^\tau = 0$. τ is a function which determines alone the electromagnetic potential. The Einstein equations written in the form (4) are convenient because we can give a direct physical interpretation of the functions involved there. 2λ is the gravitational potential and e^τ is the electromagnetic potential. If we have solved the system (4), the field equation for the k function of metric (3) is a first order differential equation

$$k_{,z} = \frac{1}{4(\ln W)_{,z}}[\frac{2W_{,zz}}{W} + (\lambda_{,z} - \gamma\tau_{,z})^2 - \gamma(\tau_{,z})^2\ e^{\lambda-(\gamma-2)\tau}] \quad (6)$$

where $z = \rho + i\,\zeta$. In terms of the functions λ and τ, the thermodynamic equation reads

$$p_{,i} + \frac{1}{2}(\lambda - \gamma\tau)_{,i}\ (p+\mu) = 0, \quad i = \rho,\ \zeta. \qquad (7)$$

System (4) is a set of three linear, second order coupled differential equations. The differential equations are coupled because the generalized Laplace operator is determined by the function W, which is determined by the pressure p and the determinant of the metric g; furthermore, the determinant of the metric g contains all the components of the metric. For the static Papapetrou line element (3) in the harmonic map parametrization, is given by $\sqrt{-g} = W\ e^{2k-\lambda+\gamma\tau}$, which contains all the unknown function of the system (4).

In what follows we will give a general exact solution of the system (4) for the thermodynamic state equation $p = \omega\mu$. We can set $\tau = 0$ in the field equation (4) and solve them and later on solve the equation for τ. We start setting $\tau = 0$. Let be $M = M(\rho,\zeta) = \frac{1}{W}$ a function restricted to

$$M(M_{,\rho\rho} + M_{,\zeta\zeta}) = (M_\rho)^2 + (M_\zeta)^2. \qquad (8)$$

In terms of M, an exact solution of system (4) with $\tau = 0$ is

$$e^\lambda = M_0 M^{-d}\ e^{\lambda_1 M}$$

$$p = \frac{M_0 M^{1-d}}{k_1}\ e^{\lambda_1 M} \qquad (9)$$

where $d = \frac{3\omega+1}{2\omega}$ and the function k_1 is given by

$$k_1 = M^{-\frac{1}{2}d^2}\exp(d\lambda_1\ M - \frac{1}{2}\lambda_1^2\ M^2) \qquad (10)$$

M_0 and λ_1 are arbitrary integration constants.

In order to have an exact solution of (2), it is necessary to fulfill the thermodynamic equation (7). Since all the functions involved in the solution depend explicitly on M, the thermodynamic equation is a differential equation on M as well. Nevertheless, all functions involved in (7) are already determined. Therefore equation (7) is a consistence equation for the two integration constants. Equation (7) will then determine the state equation of the perfect fluid we are working with. We do not have solutions for arbitrary state equations. Substituting solution (9) into (7) we obtain that $d^2 = 2$ and $\lambda_1 = 0$, this means that $\omega = -3 \pm 2\ \sqrt{2}$. The interior of this body consists of a perfect fluid which line element reads

$$ds^2 = \frac{1}{f}[\frac{M^{-\frac{1}{2}d^2}}{M^2}\Omega_M(d\rho^2 + d\zeta^2) + \frac{1}{M^2}\ d\varphi^2] - f\ dt^2,$$
$$p = fM^2, \qquad f = M_0 M^{-d}, \quad \Omega_M = M_{,\rho\rho} + M_{,\zeta\zeta} \quad (11)$$

The reader can convince himself that (11) is an exact solution of (2) and (4) by direct substitution of (11) into (2).

The Ricci scalar of metric (11) is $R = -4fM^2(3-d)$ and the rest of the invariants of this metric are powers of fM^2 or zero. Observe that the norm of the time-like Killing vector $\mathbf{X}$ vanishes if $f = 0$, $X^\nu X_\nu = f = 0$, in this region the pressure vanishes if $fM^2 = 0$. Now we have two possibilities, $d = +\sqrt{2}$ and $d = -\sqrt{2}$. For the first choice, $f \to 0$ implies $M \to \infty$, the Ricci scalar and the pressure are singular for this region and all the invariants are infinity as well. But for the second choice $d = -\sqrt{2}$ the norm of the time-like Killing vector vanishes if $M = 0$. In this region the pressure and the invariants of this metric are regular. This second metric represents the interior field of a body with a horizon. In this coordinates the whole manifold needs at least two charts, one for the interior field and at least one for the exterior field. Now the question arises; does this metric represent a black hole? The answer depends on the exterior field we match this metric with. Let me give an example.

For the exterior field we use the line element [2]

$$ds^2 = \frac{1}{f}[\frac{N^{-\frac{1}{2}d^2}}{N^3}\Omega_N(d\rho^2 + d\zeta^2) + \frac{1}{N^2}\ d\varphi^2] - f\ dt^2,$$
$$f = N_0 N^{-d} \qquad (12)$$

where now $p = \mu = 0$. It is easy to see that the metric (12) is an exact solution of the vacuum Einstein equations for arbitrary constants N_0 and d, provided that

$$N(N_{,\rho\rho}+N_{,\zeta\zeta})=2((N_\rho)^2+(N_\zeta)^2). \qquad (13)$$

Compare the equations (8) and (13), the first one is for fields inside of the object and the second one is for outside of it. One solution of (8) is $M=\frac{1}{\rho^2+\zeta^2}$ and one solution of (13) is $N=\frac{1}{\sqrt{2}\rho}$. For these solutions the line elements now read

$$\begin{aligned} ds^2 &= \frac{1}{f}[4\,M^{-1}(d\rho^2+d\zeta^2)+\frac{1}{M^2}\;d\varphi^2]-f\;dt^2,\\ p &= fM^2, \qquad f=M_0M^{\sqrt{2}}, \end{aligned} \qquad (14)$$

for inside of the body and

$$\begin{aligned} ds^2 &= \frac{1}{f}[4\,N^{-1}(d\rho^2+d\zeta^2)+\frac{1}{N^2}\;d\varphi^2]-f\;dt^2,\\ f &= N_0N^{\sqrt{2}} \end{aligned} \qquad (15)$$

for outside of it. It is now easy to match the solutions, we need only to choose $M|_R=N|_R$, where R is the boundary of the object. In what follows I investigate this region. In these coordinates ρ and ζ can take all values inside of the object provided that $\rho^2+\zeta^2\neq 0$. The region R corresponds to

$$\frac{1}{2}\frac{\sqrt{2}\left(-\sqrt{2}\rho+\rho^2+\zeta^2\right)}{\left(\rho^2+\zeta^2\right)\rho}=0 \qquad (16)$$

which has three solutions, two solutions of (16) are $\rho|_R=\frac{1}{\sqrt{2}}\pm 1/2\,\sqrt{2-4\,\zeta^2}$ and the third solution corresponds to $\rho|_R\to\infty$. Inside of the object, the pressure reads

$$p=\frac{M_0}{(\rho^2+\zeta^2)^{2+\sqrt{2}}}.$$

For the two first solutions of (16) the pressure is not necessarily small or zero. Therefore the boundary of the object corresponds to $\rho|_R\to\infty$, where $p\to 0$. A surface of constant pressure is

$$\rho^2+\zeta^2=(\frac{M_0}{p})^{\frac{1}{2+\sqrt{2}}}=r_0^2,$$

which represents a sphere of radios r_0 in the plane (ρ,ζ,φ). Strictly speaking the pressure will be zero at $\rho>>1$, but in fact this surface in not very big. Suppose that $p=10^{-6}M_0$, which could be a very small pressure, in this case the boundary surface has a radios of $r_0=7.5627$ which is not too big. Therefore the boundary surface of this object is a sphere with $p<<M_0$ and radios $r_0=(\frac{M_0}{p})^{\frac{1}{2(2+\sqrt{2})}}$.

In Boyer-Lindquist coordinates

$$\rho=\sqrt{r^2-2mr+\sigma^2}\sin\theta,$$

$$\zeta=(r-m)\cos\theta,$$

the metrics read

$$ds^2=\frac{1}{f}[\frac{1}{4}k_1\Omega_M(\frac{dr^2}{r^2-2mr+\sigma^2}+d\theta^2)+W^2\;d\varphi^2]-f\;dt^2, \qquad (17)$$

where

$$\Omega_M=M((\sqrt{r^2-2mr+\sigma^2}\;M_{,r})_{,r}\sqrt{r^2-2mr+\sigma^2}+M_{,\theta\theta})$$

and the equations (8) and (13) transform into

$$M((\sqrt{r^2-2mr+\sigma^2}\;M_{,r})_{,r}\sqrt{r^2-2mr+\sigma^2}+M_{,\theta\theta})=$$

$$n((M_{,r})^2(r^2-2mr+\sigma^2)+(M_{,\theta})^2). \qquad (18)$$

where $n=1$ corresponds to (8) and $n=2$ to (13). Using separation of variables one finds that the solutions of equation (18) are given in terms of hypergeometric functions, but this is material for further investigations [4]

Now let me magnetize this body, *i.e.*, now $\tau\neq 0$. We must solve the Maxwell equation in (4) for this particular value of λ. Nevertheless, the equation for τ in (4) should be in agreement with the integrability conditions for the electromagnetic potential A_φ in (5). But the integrability conditions for A_φ are

$$\Delta e^\tau=\Delta\tau+2M((\tau_\rho)^2+(\tau_\zeta)^2) \qquad (19)$$

which seems impossible to be fulfilled because of the equation for τ in (4), unless $\lambda-(\gamma-2)\tau=0$. This seems not to be the case because λ and τ fulfill very different differential equations. But if we choose $\gamma=2+d$ it happens a miracle. Using the solution (9) we observe that $\tau=\frac{1}{\gamma-2}\lambda=\ln(\frac{1}{M_0M})$ which implies that $\Delta e^\tau=0$. The magnetized metric now reads

$$\begin{aligned} ds^2 &= \frac{1}{f}[M^d\Omega_M(d\rho^2+d\zeta^2)+M^2d\varphi^2]-f\;dt^2\\ f &= M_1M^2\\ p &= \frac{e^{\lambda-\gamma\tau}}{M\;k_1}=M_1M^{1-d}\\ \tau &= -\frac{1}{2}\ln(2f) \end{aligned} \qquad (20)$$

where M fulfills (8). The thermodynamic equation (7) fixes again the constants. In this case we obtain $d=1$, that means $\omega=-1$ and $Q^2=\frac{1}{2M_1}$. For this values of the constants the pressure becomes constant in all of space-time and the invariants are powers of $-8M_1$ *i.e.*, constants, therefore this metric is regular all over. But now the pressure cannot vanish at all. This object is a magnetized perfect fluid with state equation $p=-\mu$, which covers the whole space-time with a constant pressure.

To obtain the space-time of a more realistic object it is necessary to study other state equations [4]. Using equations (4) and the procedure for integrating them given

here, it is possible to investigate the interior behavior of these more realistic bodies [4].

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